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The initial dispersion of contaminant in Poiseuille flow and the smoothing of the snout

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In Poiseuille flow in a circular tube passive contaminant initially spread uniformly over the cross-section would be pulled out in a paraboloidal snout in the absence of any diffusive mechanism, and there would be a discontinuity in $\bar{C}$, the mean concentration over the cross-section, associated with the contaminant at the front of the snout. In reality molecular diffusion smooths out this snout in two ways: direct longitudinal diffusion and the interaction between lateral diffusion and advection. The effect of these two mechanisms is discussed, and determined for small values of $\kappa t/a^2$, where t is the time since injection, κ is the molecular diffusivity and a is the tube radius. For such values, important in many applications, the tube walls play no part in the smoothing process. It is shown that for $\kappa t/a^2 < 0{\cdot}25(\bar{u}a/\kappa)^{-\frac{2}{3}}$, where $\bar{u}$ is the discharge velocity, the effect of longitudinal diffusion dominates over that of the interaction, which is, in turn, dominant for $\kappa t/a^2 > 2{\cdot}5(\bar{u}a/\kappa)^{-\frac{2}{3}}$, when $\bar{C}$ is close to the form described by Lighthill (1966).

1. Introduction

The dispersion of passive contaminant in flow in a tube is caused by the direct effect of molecular diffusion and by the interaction of advection and diffusion. The results of these processes for times after injection of the contaminant greater than a^2/κ, where a is a length characteristic of the dimensions of the tube cross-section and κ is the molecular diffusivity, have been investigated in a large number of papers since the pioneering work by Taylor (1953).

However in many important flows, a^2/κ, the time taken for a molecule of contaminant to wander over the tube cross-section, is much greater than the time taken for it to be carried right through the tube. Thus in the human aorta $a \approx 1$ cm, so that $a^2/\kappa \approx 10^5$ s, yet a fluid particle travels right through the aorta in several seconds. In smaller arteries the difference between the two time scales is less marked as Lighthill (1966) pointed out, but it is in the large arteries that most of the important work with injected contaminants occurs.

Thus it is important to know how the dispersion of an injected contaminant proceeds for times t after injection such that $\kappa t/a^2 \ll 1$. This is also an interesting problem theoretically, since nearly all previous work (e.g. Chatwin 1970) on the dispersion process has given results valid asymptotically as $\kappa t/a^2 \to \infty$, but not valid as $\kappa t/a^2 \to 0$, and there is clearly a need to match such results to others valid for small times. Technically it is more difficult to obtain results for small times than for large times. Not only does the diffusion equation have a singularity at $t = 0$, but also the details of the flow and of the initial distribution of contaminant affect the dispersion process more markedly for small times than for large times.

Lighthill (1966) was however able to obtain some important results for the particular case of Poiseuille flow in a circular tube of radius a, when C, the distribution of concentration, satisfies

$$\frac{\partial C}{\partial t} + 2\overline{u}\left(1 - \frac{r^2}{a^2}\right)\frac{\partial C}{dx} = \kappa \nabla^2 C, \tag{1.1}$$

where $\overline{u}$ is the discharge velocity, x measures distance along the tube axis and r is the radial co-ordinate. Lighthill considered only the case when the contaminant is initially distributed uniformly over the cross-section, so that

$$C = f(x) \text{ at } t = 0. \tag{1.2}$$

In the absence of any diffusion whatsoever the contaminant near the tube axis is pulled forward faster than that nearer the walls, so that the forward portion of the cloud of contaminant forms a paraboloidal snout with sharp edges. But in reality diffusion smooths out this snout by two separate mechanisms: direct longitudinal diffusion and interaction between advection and cross-sectional diffusion. Lighthill considered only the second of these, whereas the purpose of the present paper is to extend Lighthill's results to include the effects of both mechanisms.

In order to appreciate that both effects can be important it is useful to consider the special initial distribution when

$$\overline{C} = \delta(x) \text{ at } t = 0. \tag{1.3}$$

In the absence of diffusion $C(x, r, t)$ satisfies

$$C(x,r,t) = (\pi a^2)^{-1}\delta[x - 2\overline{u}t(1 - r^2/a^2)], \tag{1.4}$$

and this is plotted schematically in figure 1, as is the value of $\overline{C}(x,t)$, the mean concentration over the cross-section, which, using (1.4), is given by

$$\overline{C}(x,t) = \begin{Bmatrix} (2\overline{u}t)^{-1} & \text{for } 0 < x < 2\overline{u}t, \\ 0 & \text{otherwise.} \end{Bmatrix} \tag{1.5}$$

(Throughout this paper the units of concentration will be chosen such that

$$\int_{-\infty}^{\infty} \overline{C}\, dx = 1.$$

This is for algebraic simplicity and is possible since all equations are linear in C.) The results (1.4) and (1.5) are given by Taylor (1953). As explained above, the present paper examines how the discontinuity at $x = 2\overline{u}t$ is smoothed out. The direct effect of longitudinal diffusion smooths out the discontinuity over an axial distance of order $(\kappa t)^{\frac{1}{2}}$. On the other hand, lateral diffusion causes material

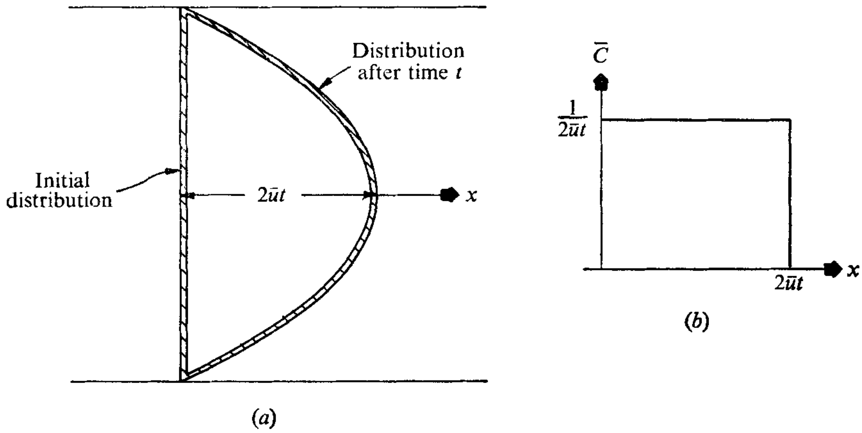

FIGURE 1. (*a*) The effect of advection alone on a distribution of concentration initially uniform over the cross-section. (*b*) The mean concentration over the cross-section for the distribution sketched in (*a*).

initially on the axis to be spread laterally over a distance of the same order, and thus to have an axial velocity of order $2\bar{u}\kappa t/a^2$ *less* than it would have had it remained on the axis. Hence this mechanism leads to a smoothing out of the discontinuity over an axial distance of order $\bar{u}\kappa t^2/a^2$, as Lighthill (1966) found. Another difference between the two smoothing mechanisms is also apparent, for longitudinal diffusion causes *half* of the material initially near the axis to have an axial displacement *greater* than $2\bar{u}t$, but the interaction mechanism acting alone does not permit *any* material to have an axial displacement greater than $2\bar{u}t$ (as illustrated in figure 2 of Lighthill's paper).

The preceding argument, and the analysis in the remainder of the paper, ignores all effects of the tube walls, so that it is supposed throughout this paper that $\kappa t/a^2 \ll 1$. Now $(\kappa t)^{\frac{1}{2}} \approx \kappa\bar{u}t^2/a^2$ when $\kappa t/a^2 \approx (\bar{u}a/\kappa)^{-\frac{2}{3}}$, so in the common practical situation with $\bar{u}a/\kappa \gg 1$ the interaction mechanism described by Lighthill is the predominant one for times such that $(\bar{u}a/\kappa)^{-\frac{2}{3}} \ll \kappa t/a^2 \ll 1$. But in all flows, the longitudinal diffusion mechanism is the predominant one for times such that $\kappa t/a^2$ is much less than the smaller of $(\bar{u}a/\kappa)^{-\frac{2}{3}}$ and 1. Granted the difference in type between the two smoothing mechanisms it is important to see how one is replaced by the other. Note also that in the uncommon flows for which $\bar{u}a/\kappa$ is not much greater than 1, the interaction mechanism which replaces longitudinal diffusion as time increases involves the walls, and so is essentially of the form described by Taylor (1953) rather than that described by Lighthill (1966).

The remainder of the paper is concerned with making these rather general arguments precise. In § 2 a direct extension of Lighthill's (1966) analysis is examined, but this applies only in Poiseuille flow. In § 3 results consistent with those in § 2 are obtained by another method which could be applied to flows other than Poiseuille flow.

2. An exact solution

It is clear from the preceding arguments that the walls play no part in the smoothing of the front of the paraboloidal snout for times t after injection such that $\kappa t/a^2 \ll 1$. Lighthill (1966) examines this in detail and his arguments should apply even when longitudinal diffusion is included for this does not seem likely to increase radial derivatives of C. As a consequence the smoothing of the snout can be analysed by supposing that (1.1) and (1.2) hold throughout space. In addition the proper boundary condition of zero flux across the tube walls can be replaced by the condition

$$C \to 0 \quad \text{as} \quad r \to \infty. \tag{2.1}$$

Now (1.1) has the exact solution

$$C = \exp\{ik(x-2\bar{u}t)-(\alpha r^2/4\kappa)\tanh\alpha t-\kappa k^2 t\}\operatorname{sech}\alpha t, \tag{2.2}$$

where

$$\alpha^2 = -8\bar{u}ik\kappa/a^2. \tag{2.3}$$

Then, provided axes are chosen such that $f(x)$ in (1.2) is zero for $x > 0$, the solution of (1.1) which satisfies (1.2) and (2.1) is (as in Lighthill 1966)

$$C = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} F(k)\operatorname{sech}\alpha t\left\{\exp\left[ik(x-2\bar{u}t)-\frac{\alpha r^2}{4\kappa}\tanh\alpha t-\kappa k^2 t\right]\right\}dk, \tag{2.4}$$

where $\epsilon > 0$, and $F(k)$ is the Fourier transform of $f(x)$, so that

$$F(k) = \int_{-\infty}^{0} f(x)\,e^{-ikx}\,dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} F(k)\,e^{ikx}\,dk. \tag{2.5}$$

Also, the value of the mean concentration $\bar{C}(x,t)$ is given by integrating $r\bar{C}$ [with C given in (2.4)] over all r from 0 to ∞ (since the tube walls are ignored) and dividing by $\frac{1}{2}a^2$. This yields

$$\bar{C}(x,t) = \frac{2\kappa t}{\pi a^2}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} F(k)\frac{\exp[ik(x-2\bar{u}t)-\kappa k^2 t]}{\alpha t\sinh\alpha t}\,dk. \tag{2.6}$$

The results (2.2), (2.4) and (2.6) differ from the corresponding results in Lighthill (1966) by having the term $\exp[-\kappa k^2 t]$. This comes from the longitudinal diffusion term in (1.1), i.e. $\kappa\,\partial^2 C/\partial x^2$, which Lighthill neglected. However the presence of this term makes it impossible to handle (2.4) and (2.6) in the way Lighthill did, for $\exp[-\kappa k^2 t]$ takes large values on parts of each of the semicircles $k = Ke^{i\theta}$ ($K > 0$, $0 < \theta < \pi$) and $k = Ke^{i\theta}$ ($K > 0$, $-\pi < \theta < 0$). These were the contours used by Lighthill.

The subsequent discussion in this section shows how the expression (2.6) for $\bar{C}$ can be handled. Writing $s = \alpha t$ and using (2.3) gives

$$\bar{C} = \frac{i}{2\pi\bar{u}t}\int_{C_1} F\left(\frac{is^2a^2}{8\bar{u}\kappa t^2}\right)\frac{\exp[Ms^4+Ns^2]}{\sinh s}\,ds, \tag{2.7}$$

where the contour C_1 is illustrated in figure 2 and

$$M = \frac{a^4}{64\bar{u}^2\kappa t^3}, \quad N = \frac{a^2}{4\kappa t}\left(1-\frac{x}{2\bar{u}t}\right). \tag{2.8}$$

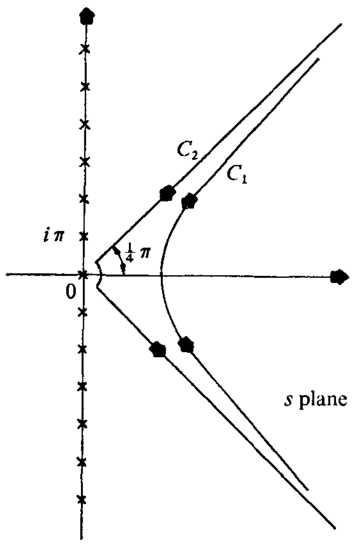

FIGURE 2. The contours used in the determination of $\bar{C}$.

Now M is always positive while N may be positive or negative. Also

$$64M = [(\kappa t)^{\frac{1}{2}}/\kappa \bar{u} t^2/a^2]^2, \tag{2.9}$$

so that, essentially, M is the square of the ratio of the distance over which longitudinal diffusion smooths out the discontinuity to that over which the interaction mechanism smooths it out. As t increases from 0 to ∞, M decreases from ∞ to 0, and Lighthill's result is obtained by setting $M = 0$.

Now arbitrary initial distributions of the form (1.2) can be built up by superposition from those of the form (1.3), for which, using (2.5) and (2.7),

$$\bar{C} = \frac{i}{2\pi \bar{u} t} \int_{C_1} \frac{\exp[Ms^4 + Ns^2]}{\sinh s} ds. \tag{2.10}$$

The only singularities of the integrand are on the imaginary axis at $s = 0, \pm i\pi, \pm 2i\pi, \ldots$. Thus the integral over the contour C_1 can be replaced by one over the contour C_2, also shown in figure 2. The contour C_2 consists of two straight lines and a quarter-circle. The integral is independent of the radius δ of the quarter-circle, so it is convenient to consider δ to be small. An easy analysis shows that the quarter-circle contributes an amount $[(4\bar{u}t)^{-1} + O(\delta)]$ to (2.10), so that

$$\bar{C} = \frac{1}{4\bar{u}t} + \frac{1}{4\pi \bar{u} t} \lim_{\delta \to 0} \left[\int_{\delta^{\frac{1}{2}}}^{\infty} \frac{\exp[-M\sigma^2 - iN\sigma]\, d\sigma}{\sigma^{\frac{1}{2}} e^{-\frac{1}{4}i\pi} \sinh(\sigma^{\frac{1}{2}} e^{-\frac{1}{4}i\pi})} + \text{c.c.} \right], \tag{2.11}$$

where c.c. stands for complex conjugate.

Now (2.11) can be approximately evaluated in some special cases. Near $\sigma = 0$,

$$\frac{1}{\sigma^{\frac{1}{2}} e^{-\frac{1}{4}i\pi} \sinh(\sigma^{\frac{1}{2}} e^{-\frac{1}{4}i\pi})} \approx i\left[\frac{1}{\sigma} - \frac{7\sigma}{360} + \ldots\right] - \left[\frac{1}{6} - \frac{31\sigma^2}{15120} + \ldots\right]. \tag{2.12}$$

N \ M	10	1	0·1	0·01	0·001	Lighthill ($M = 0$)
$-\infty$	0·00	0·00	0·00	0·00	0·00	0·00
-5	0·12	0·00	0·00	0·00	0·00	0·00
-1	0·40	0·21	0·02	0·00	0·00	0·00
$-\frac{1}{2}$	0·44	0·32	0·07	0·00	0·00	0·00
$-\frac{1}{4}$	0·46	0·38	0·18	0·00	0·00	0·00
$-\frac{1}{8}$	0·47	0·42	0·26	0·04	0·00	0·00
$-\frac{1}{16}$	0·48	0·44	0·31	0·09	0·00	0·00
0	0·49	0·45	0·36	0·17	0·02	0·00
$\frac{1}{16}$	0·49	0·47	0·41	0·28	0·15	0·08
$\frac{1}{8}$	0·50	0·49	0·47	0·42	0·41	0·43
$\frac{1}{4}$	0·51	0·52	0·57	0·70	0·81	0·83
$\frac{1}{2}$	0·53	0·59	0·77	0·96	0·98	0·99
1	0·57	0·72	0·99	1·00	1·00	1·00
5	0·86	1·00	1·00	1·00	1·00	1·00
∞	1·00	1·00	1·00	1·00	1·00	1·00

TABLE 1. Values of $2\bar{u}t\bar{C}$ from (2.14) for various values of $M = a^4/64\bar{u}^2\kappa t^3$ and $N = (a^2/4\kappa t)(1-x/2\bar{u}t)$. Also tabulated is Lighthill's (1966) result given in (2.15)

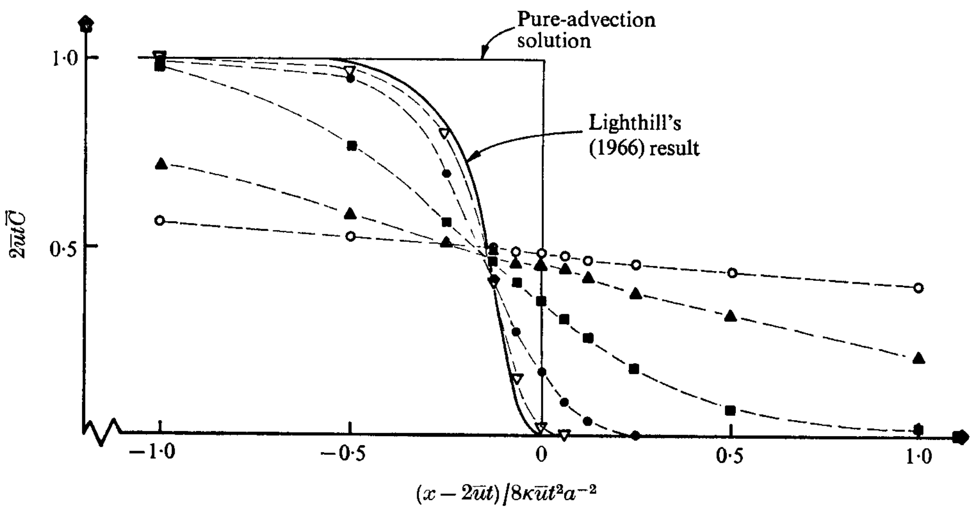

FIGURE 3. Graphs of $\bar{C}$ for various values of M compared with Lighthill's (1966) result and with the pure-advection solution shown in figure 1. ○, $M = 10$; ▲, $M = 1$; ■, $M = 0{\cdot}1$; ●, $M = 0{\cdot}01$; ▽, $M = 0{\cdot}001$.

For $M \gg 1$, $\exp(-M\sigma^2)$ decreases rapidly as σ increases, so that an argument similar to that used in the method of steepest descent can be used to evaluate $\bar{C}$ asymptotically, using the expansion (2.12) and the following integrals:

$$\int_0^\infty \exp(-M\sigma^2)\frac{\sin N\sigma}{\sigma}d\sigma = \tfrac{1}{2}\pi\,\mathrm{erf}\left[\frac{N}{2M^{\frac{1}{2}}}\right],$$

$$\int_0^\infty \exp(-M\sigma^2)\cos N\sigma\,d\sigma = \frac{\pi^{\frac{1}{2}}}{2M^{\frac{1}{2}}}\exp\left[-\frac{N^2}{4M}\right],$$

etc., where each member of the series (except the first) is obtained by differentiating the previous member with respect to N. Hence on integrating term by term,

$$\bar{C} \approx \frac{1}{4\bar{u}t}\left\{1+\operatorname{erf}\left[\frac{N}{2M^{\frac{1}{2}}}\right]\right\} - \frac{\exp\left[-N^2/4M\right]}{24\bar{u}t(\pi M)^{\frac{1}{2}}}$$
$$\times\left\{1+\frac{7}{60M^{\frac{1}{2}}}\left[\frac{N}{2M^{\frac{1}{2}}}\right]-\frac{31}{5040M}\left(1-2\left[\frac{N}{2M^{\frac{1}{2}}}\right]^2\right)\ldots\right\}, \quad (2.13)$$

which is valid asymptotically as $M \to \infty$ for $N/2M^{\frac{1}{2}}$ of order 1 or less. It is also easy to show by the method of stationary phase that (2.13) holds for $|N/2M^{\frac{1}{2}}| \gg 1$, when $\exp\left[-N^2/4M\right]$ is negligibly small.

For other values of N and M the integral for $\bar{C}$ has to be evaluated numerically. This can conveniently be done by substituting $\sigma = 2\tau^2$ in (2.11), expressing the integrand in terms of its real and imaginary parts and rearranging, when the following result is obtained:

$$\bar{C} = \frac{1}{4\bar{u}t}\left\{1+\operatorname{erf}\left[\frac{N}{2M^{\frac{1}{2}}}\right]\right\} - \frac{1}{\pi\bar{u}t}\{I+J\}. \quad (2.14)$$

Here

$$I = \int_0^\infty \exp\left(-4M\tau^4\right)\cos 2N\tau^2\left[\frac{\cosh\tau\sin\tau-\sinh\tau\cos\tau}{\sinh^2\tau+\sin^2\tau}\right]d\tau,$$
$$J = \int_0^\infty \exp\left(-4M\tau^4\right)\sin 2N\tau^2\left[\frac{1}{\tau}-\frac{\cosh\tau\sin\tau+\sinh\tau\cos\tau}{\sinh^2\tau+\sin^2\tau}\right]d\tau.$$

There is no difficulty in the calculations and some results are given in table 1. Also given in the table is Lighthill's (1966) result, which can be written for $N > 0$ in each of the following ways:

$$\bar{C} = \frac{1}{2\bar{u}t}+\frac{1}{\bar{u}t}\sum_{n=1}^{\infty}(-1)^n\exp\left[-n^2\pi^2N\right]$$
$$= \left(\frac{4}{\pi N}\right)^{\frac{1}{2}}\sum_{n=0}^{\infty}\exp\left[-(2n+1)^2/4N\right]. \quad (2.15)$$

For $N < 0$, i.e. for $x > 2\bar{u}t$, Lighthill naturally found $\bar{C} = 0$, since direct longitudinal diffusion is the only mechanism which can transport contaminant molecules faster than the maximum fluid velocty.

The results in table 1 are plotted in figure 3† to show how Lighthill's result (2.15) is approached as M decreases, i.e. as t increases. In practice Lighthill's result holds for $M < 0{\cdot}001$, i.e. for

$$\kappa t/a^2 > 2{\cdot}5(\bar{u}a/\kappa)^{-\frac{2}{3}}, \quad (2.16)$$

using (2.8). This result is consistent with the arguments in § 1. For smaller values of $\kappa t/a^2$ the direct effect of longitudinal diffusion is important, and indeed

† In figure 3 the axial co-ordinate is stretched by a factor proportional to t^{-2} to facilitate comparison with (2.15) for *large t* (*small M*), so that the distance over which longitudinal diffusion smooths the snout for *small t* (*large M*) appears larger than it is. Some of the results are plotted with a different scaling in figure 4.

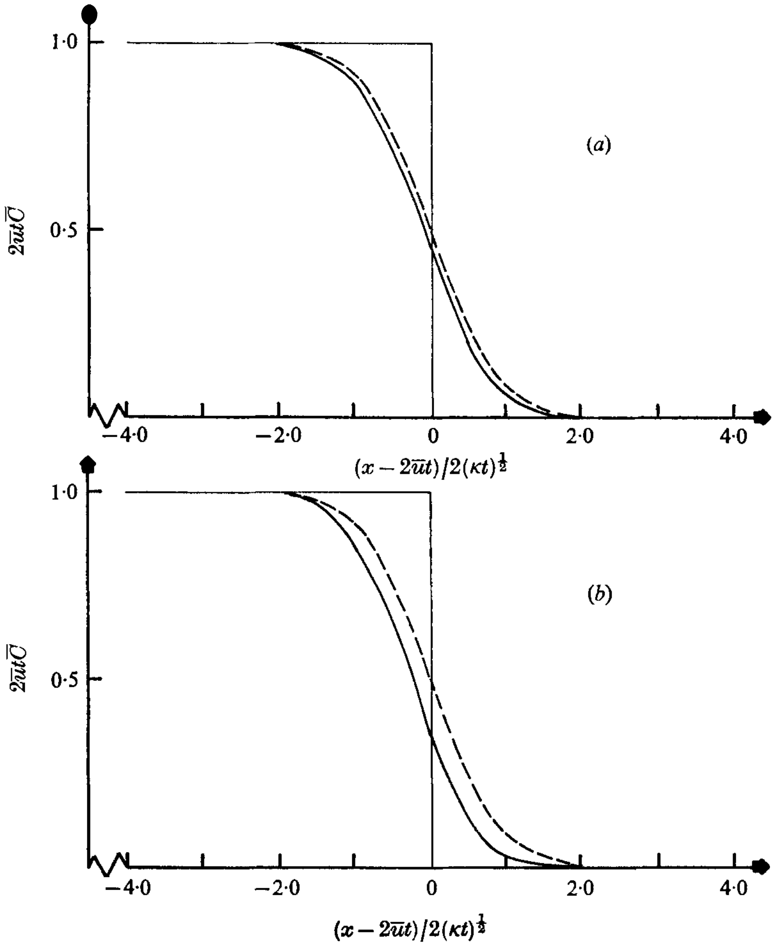

FIGURE 4. Graphs of $\bar{C}$ for $M = 1$ and $M = 0{\cdot}1$ (solid lines) compared with $(4\bar{u}t)^{-1}$ $\{1+\operatorname{erf}[N/2M^{\frac{1}{2}}]\}$ (broken lines), the first term of the asymptotic expansion (2.13). (a) $M = 1$. (b) $M = 0{\cdot}1$.

predominates for $M > 0{\cdot}1$ as figure 4 shows. Particularly significant is the value of $\bar{C}$ at $x = 2\bar{u}t$, which, for $M > 1$, is almost $(4\bar{u}t)^{-1}$, i.e. one-half of the value taken in $0 < x < 2\bar{u}t$ by the pure-advection theory (1.5). This is an expected result, as explained in § 1. Figure 4 shows that for $M > 1$, i.e. for

$$\kappa t/a^2 < 0{\cdot}25(\bar{u}a/\kappa)^{-\frac{2}{3}}, \qquad (2.17)$$

the form of $\bar{C}$ is given by

$$\bar{C} \approx (4\bar{u}t)^{-1}\{1+\operatorname{erf}[N/2M^{\frac{1}{2}}]\},$$

i.e. by the first term of the asymptotic solution (2.13).

Since the analysis of this section holds only if the contaminant does not interact with the tube walls, the estimates (2.16) and (2.17) must be supplemented by the further condition $\kappa t/a^2 \lesssim 0{\cdot}1$, as Lighthill (1966) showed.

Although the detailed results of this section cannot be applied to the complicated, usually unsteady, flows that frequently occur in practice, the general arguments of § 1 and the estimates (2.16) and (2.17) should have some validity. Consider for example the human aorta, where $\bar{u}a/\kappa \approx 10^6$, based on the mean discharge velocity, so that $(\bar{u}a/\kappa)^{-\frac{2}{3}}$, the parameter appearing in (2.16) and (2.17), is of order 10^{-4}. Now the time taken for a molecule moving with the mean discharge velocity to pass right along the aorta is of the order of 5 s, giving a maximum possible value of $\kappa t/a^2$ of order 5×10^{-5}. It then appears from (2.16) and (2.17) that for most of the time that the contaminant remains in the aorta the effect of direct longitudinal diffusion is much greater than that of the interaction considered by Lighthill. But this conclusion remains tentative without calculations using more realistic velocity profiles.

3. Another approach

The merit of the approach to be described in this section is that it can be applied to flows other than Poiseuille flow although in this paper only Poiseuille flow will be considered in detail. It is hoped to deal in detail with other flows, especially those which are not unidirectional, in a later paper.

Consider contaminant released from $\mathbf{r}_0$ at $t = 0$ in a steady flow with velocity $\mathbf{u}(\mathbf{r})$, so that the concentration C satisfies

$$\partial C/\partial t + \mathbf{u}\,.\,\nabla C = \kappa \nabla^2 C \tag{3.1}$$

and

$$C \propto \delta(\mathbf{r}-\mathbf{r}_0) \quad \text{at} \quad t = 0. \tag{3.2}$$

Retaining the assumption of the earlier sections of this paper that the tube walls play no part in the dispersion process, the approximate solution of (3.1) and (3.2) for sufficiently small times is (see, for example, Saffman 1960)

$$C \approx \frac{A}{8(\pi\kappa t)^{\frac{3}{2}}} \exp\left[-\mathbf{R}^2\right], \tag{3.3}$$

where A is a constant to be chosen later, and

$$\mathbf{R} = \{\mathbf{r}-\mathbf{r}_0 - t\mathbf{u}(\mathbf{r}_0)\}/2(\kappa t)^{\frac{1}{2}} = (X, Y, Z). \tag{3.4}$$

Thus the cloud of contaminant initially spreads isotropically, entirely as the result of diffusion, about the fluid particle initially coincident with the source. Now it is easy to see that (3.1) has a formal solution, with (3.3) as its leading term, of the form

$$C = \frac{A}{8(\pi\kappa t)^{\frac{3}{2}}} \exp\left[-\mathbf{R}^2\right]\left\{1 + \sum_{n=1}^{\infty} T^n\,\Gamma_n(\mathbf{R})\right\}, \tag{3.5}$$

where $\mathbf{R}$ is defined in (3.4) and

$$T = (\kappa t/a^2)^{\frac{1}{2}}. \tag{3.6}$$

Equations for the Γ_n can be obtained by substituting (3.5) into (3.1), expanding $\mathbf{u}(\mathbf{r})$ in a Taylor series about $\mathbf{r}_0$ and equating coefficients of like powers of T. The equations are solved subject to the boundary condition

$$\Gamma_n \exp\left[-\mathbf{R}^2\right] \to 0 \quad \text{as} \quad |\mathbf{R}| \to \infty, \tag{3.7}$$

thus ensuring that each term of the expansion (3.5) tends to zero as $|\mathbf{R}| \to \infty$, in accordance with (2.1).

When the flow is Poiseuille flow in a circular tube – the subject of this paper – then

$$\mathbf{u}(\mathbf{r}) = (2\bar{u}\{1 - y^2/a^2 - z^2/a^2\}, 0, 0)$$

and the equations for the Γ_n have simple polynomial solutions. In particular, taking

$$\mathbf{r}_0 = (0, y_0, z_0), \tag{3.8}$$

it is found that

$$\left.\begin{aligned} \Gamma_1 &= 0, \\ \Gamma_2 &= -(4\bar{u}/\kappa)X[y_0 Y + z_0 Z], \\ \Gamma_3 &= -\tfrac{4}{3}(\bar{u}a/\kappa)X[2Y^2 + 2Z^2 + 1], \\ \Gamma_4 &= -\tfrac{8}{3}(\bar{u}/\kappa)^2[1 - 2X^2][6(y_0 Y + z_0 Z)^2 + (y_0^2 + z_0^2)], \end{aligned}\right\} \tag{3.9}$$

where X, Y and Z are defined in (3.4).

In order to link this result in with the results of § 2, the value of $\bar{C}$ associated with the C in (3.5) is obtained in the way described immediately preceding (2.6). For the case of Poiseuille flow, when (3.9) hold, the result is

$$\bar{C} = \frac{A}{\pi a^2}\frac{\exp[-X^2]}{2(\pi\kappa t)^{\frac{1}{2}}}\left[1 - \frac{4\bar{u}a}{\kappa}T^3X + \frac{8}{3}\left(\frac{\bar{u}}{\kappa}\right)^2(y_0^2 + z_0^2)T^4(2X^2 - 1)\ldots\right]. \tag{3.10}$$

Finally (3.9) must be averaged over all possible source points so that the initial condition (1.3), on which the detailed results of § 2 are based, can be met. Hence the value of $\bar{C}$ in (3.10) is integrated over all y_0 and z_0 [note that X depends on y_0 and z_0 because of its dependence on $\mathbf{u}(\mathbf{r}_0)$ shown in (3.4)] and A is chosen such that

$$\int_{-\infty}^{\infty} \bar{C}\,dx = 1,$$

as it was in §§ 1 and 2. The result is

$$\bar{C} = \frac{1}{4\bar{u}t}\left\{1 + \operatorname{erf}\left[\frac{N}{2M^{\frac{1}{2}}}\right]\right\} - \frac{\exp[-N^2/4M]}{24\bar{u}t(\pi M)^{\frac{1}{2}}} + \ldots,$$

where M and N are as in § 2. This consists of the first two terms of the asymptotic expansion given in (2.13), so the analysis in this section is consistent with § 2, when the initial distribution is uniform over the cross-section.

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